Hierarchical Power Delivery Network Analysis Using Markov Chains

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Abstract—This paper proposes a Markov chain based hierarchical method to efficiently analyze the power delivery network. After the network being partitioned into several subnetworks, each subnetwork is transformed into a local Markov chain. Then, the connective relations between all subnetworks are modeled as a global Markov chain. Finally, those local and the global Markov chains are incorporated to build a hierarchical bipartite Markov chain engine to analyze the power delivery network.

The experimental results not only demonstrate the accuracy of proposed method compared with a very accurate time domain solver [1], but also show its significant runtime improvement, over 200 times faster than the InductWise [1] and over 10 times faster than the IEKS method [2], and less memory usage.

I. INTRODUCTION

In the consecutive technology generations, different chips’ characteristics, such as higher operating frequencies, larger number of transistors and lower supply voltages, have made the integrity issues of power delivery network become key factors for high performance designs [3]. The quality of power delivery systems can be degraded by many different noises, such as IR drop, Ldi/dt drop, and resonance fluctuations. Because of the differentiation nature of Ldi/dt drop, the extensive transient simulations are necessary to ensure that the power delivery design meets the performance target and reliability goal. Due to the tremendous amount of circuit elements in the power distribution network, the direct solvers, such as SPICE [4], require long runtime and consume huge memory.

Several approaches have been developed to speed up the power delivery network analysis, for instance, multi-grid based methods [6], hierarchical methods [2], [5], and stochastic based approaches [7], [8]. Our hierarchical solver is inspired by the random walk like methods [7], [8]. The random walk based framework was developed in [7] to reduce the memory usage and runtime, and the absorbing Markov chain based method was introduced in [8]. The main advantage of the absorbing Markov chain method over random walk is that it converges faster than the random walk method because of its parallel walks property. However, the transition probability matrix is getting denser as the symbolic relaxation procedure [8] proceeds, and such dense matrix actually increases the computational cost and memory usage.

To develop a more efficiently iterative method with sparse matrices and high convergent rate in transient analysis, we present a hierarchical power delivery network analysis engine by using the bipartite Markov chains, a global Markov chain and multiple local Markov chains. First, the power delivery network is partitioned into several subnetworks. After that, each subnetwork is modeled as a local Markov chain. Then, the connectivities between all subnetworks are modeled as a global Markov chain. Finally, the local and global Markov chains are integrated to build a hierarchical bipartite Markov chain engine which we name it as “PDABiMaC”. The main advantage of “PDABiMaC” is that it partially finds the limit behavior of local and global Markov chains by using the strategy of divide and conquer to build the bipartite Markov chain recurrence. Once the bipartite Markov chain recurrence has been built, the transient simulation can be very efficient because of its fast convergence.

The rest of paper is organized as follows. Section II introduces the absorbing Markov chain model for the RC (resistance-capacitance) power delivery network analysis, and the proposed algorithm, PDABiMaC, for RC circuits is presented in Section III. Then, the effects of mutual inductances are taken into account in Sections IV. Finally, the experimental results and conclusion are shown in Section V and VI.

II. ABSORBING MARKOV CHAIN MODEL ON RC POWER DELIVERY NETWORK ANALYSIS

Without considering the inductance effects, the power delivery network can be modeled as an equivalent RC network. By applying the KCL (Kirchoff’s current law) and the trapezoidal approximation with time step h at an arbitrary node x of this equivalent RC network, its sampling voltage at time t, \( v_x^t \), is

\[ v_x^t = \sum_{j \in \mathcal{d}(x)} g_{xj} \cdot v_{j}^{t-1} + \frac{2c_x}{h} \cdot v_x^{t-1} \cdot h + f_x^t, \quad (1) \]

where \( f_x^t = - \sum_{j \in \mathcal{d}(x)} \frac{g_{xj}}{g_x} \cdot v_x^{t-1} - \frac{1}{g_x} \cdot v_x^{t-1} \cdot h - \frac{1}{g_x} \cdot v_x^{t-1} \cdot h', \quad (2) \)

the superscript and the suffix of each voltage, \( v \), stand for the sampling time and the sampling node respectively, \( \mathcal{d}(x) \) is the set of nodes adjacent to node \( x \); \( g_{xj} \) is the conductance between node \( x \) and its adjacent node \( j \); \( c_x \) is the gate and decoupling capacitances at node \( x \); \( g_x \) is equal to \( \sum_{j \in \mathcal{d}(x)} g_{xj} \cdot 2c_x/h \); and \( v_x^{t-1} \) and \( v_x^{t-1} \) are the sampling values at time \( t-1 \) and \( t \) of the independent current source attached at node \( x \). Since the summation of \( \sum_{j \in \mathcal{d}(x)} \frac{g_{xj}}{g_x} \cdot 2c_x/h \) in Equation (1) is equal to one, the set of modified nodal equations can be translated to a Markov chain by setting nodal voltages as states’ variables and their corresponding coefficients in each equation as transition probabilities. Hence, Equation (1) can be rewritten as

\[ v_x^t = \sum_{j \in \mathcal{d}(x)} p_{xj} v_j^t + p_{x x_{lst}} v_{x_{lst}}^{t-1} + f_x^t, \quad (3) \]

where \( x_{lst} \) is regarding to the last sampling state variable at node \( x \); \( p_{xj} \) and \( p_{x x_{lst}} \) are transition probabilities from node \( x \) to node \( j \) and \( x_{lst} \) respectively, \( p_{xj} = g_{xj} / g_x \); \( p_{x x_{lst}} = 2c_x / (h \cdot g_x) \), and \( f_x^t \) is a known value at time \( t \).

To illustrate the meaning of Equation (3), a portion of equivalent RC circuit shown in Fig. 1.(a) can be transformed
to its corresponding Markov chain state diagram as shown in Fig. 1(b). Because $f^e_j$ is a known value at the sampling time $t$ and for simplicity, it is not shown in Fig. 1.(b). Since the procedure of random walk is only necessary to be performed on the unknown nodal voltage state and $v_{j}^{t-h}$ is a known value at time $t$, the transition probability from state $v_{j}^{t-h}$ to state $v_{j}^{t}$ is zero. With the same reason, the outward probability of each supply voltage is also equal to zero. Those states with zero outward probabilities are called absorbing states [9].

With utilizing the formal expression of finite absorbing Markov chain [9] and above description, all unknown nodal voltages can be written in a matrix form as follows.

$$V^t_{x} = RV^t_{V} + QV^t_{V} + PV^t_{x-h} + f^x_{t}$$

where $V^t_{V}$, $V^t_{V}$ and $V^t_{x-h}$ are state vectors of supply voltages, unknown nodal voltages at time $t$, and nodal voltages at time $t-h$, respectively. $Q$, $R$ and $P$ are transition probability matrices between unknown nodal voltage states, from unknown nodal voltage states to supply voltage states, and from unknown nodal voltage states to their last sampling nodal voltage states, respectively. The $f^x_{t}$ is a known valued vector at time $t$.

With utilizing the Markov chain relaxation [9], $V^t_{x}$ can be approximated by the following recurrence.

$$V^{*t}_{x} \approx V^t_{x} = Q^{n}RV^t_{V} + Q^{n}PV^t_{x-h} + Q^{n}PV^{x-h} + Q^{n}PV^{3-h} + \ldots + Q^{n}PV^{t-h} + Q^{n}PV^{t-h} + f^x_{t}$$

where $n$ is the number of Markov chain relaxation.

The sum of entries for each row of $Q$ is bounded by one, and the absolute value of each eigenvalue of $Q$ can be proved to be less than or equal to one by Gershgorin’s Disk Theorem [10]. Therefore, the convergent property of Equation (5) outperforms the convergent property of directly applying the relaxation procedure to the classical MNA (modifide nodal analysis) matrix for computing $V^t_{x}$.

To further study the property of finite absorbing Markov chain, $V^t_{x}$ can be obtained as $n$ goes to infinity.

$$V^{t}_{x} = B_{x\rightarrow V}V^t_{V} + B_{x\rightarrow x_{lst}}V^{t-h} + Nf^x_{t}$$

where $N = \sum_{m=0}^{\infty}Q^{m} = (I - Q)^{-1}$, $B_{x\rightarrow V} = NR$, and $B_{x\rightarrow x_{lst}} = NP$

$N$ is a fundamental matrix, and its entry $n_{ij}$ is the expected time that the chain meets state $v_{i}^{t}$ before absorption as it starts in state $v_{j}^{t}$. The $B_{x\rightarrow V}$ and $B_{x\rightarrow x_{lst}}$ are the absorption probability matrices, and their entries indicate the probabilities from the unknown nodal voltage states to the supply voltage states, and the last sampling voltage states, respectively.

Those unknown nodal voltage states are viewed as internal node states, and the supply voltage states and last sampling nodal voltage states are viewed as absorbing states. Although Equation (6) is not directly used in our implementation, it indicates that the exact value of each internal node state is equal to the weighted average of absorbing states minus the average voltage drop caused by independent current sources.

III. HIERARCHICAL ALGORITHM ON POWER DELIVERY NETWORK ANALYSIS

In this section, the strategy of divide and conquer is utilized to speed up the analysis and save the memory usage. The framework of our PDABioMaC for the RC network is summarized in Fig. 3. Each step will be discussed in the following subsections. To simplify the expression, several terms and symbols are defined in Fig. 2 and the sketch map of local and global Markov chain is shown in Fig. 4.

Algorithm: PDABioMaC

A1. Partition the given network into multiple subnetworks.
A2. Construct the bipartite Markov chain model of the given network.
A2.1 Build a local Markov chain for each subnetwork to characterize the connectivities of its internal nodes and the nodes on its adjacent cut lines.
A2.2 Set up a global Markov chain to describe the connectivities of all nodes on cut lines and their adjacent nodes situated in each subnetwork.
A3. Solve all unknown nodal voltages of the given network by using the proposed "bipartite absorbing Markov chain relaxation" method.

B. Equivalent Local Markov Chain Model for Each Subnetwork

After partitioning, the equivalent local Markov chain model of each subnetwork can be constructed by performing the absorbing Markov chain relaxation presented in Section II to each subnetwork. For example, the nodal voltages of internal nodes in each $sub_j$ can be represented as

$$V^{t}_{x_{sub_j}} = B_{x_{sub_j}\rightarrow V}V^t_{V_{sub_j}} + B_{x_{sub_j}\rightarrow x_{lst_{sub_j}}}V^{t-h}_{x_{sub_j}} + N_{x_{sub_j}}x_{sub_j}f_{x_{sub_j}}^{t}$$

where $V^t_{V_{sub_j}}$, $V^{t-h}_{x_{sub_j}}$ and $V^{t-h}_{x_{sub_j}}$ are state vectors of $x_{sub_j}$ at time $t$, $x_{sub_j}$ at time $t-h$, unknown nodal voltages of $x_{sub_j}$, and $V^{t-h}_{x_{sub_j}}$, respectively. $B_{x_{sub_j}\rightarrow V}$ and $B_{x_{sub_j}\rightarrow x_{lst_{sub_j}}}$ and $B_{x_{sub_j}\rightarrow V}$ are absorption probability matrices from $x_{sub_j}$ to $V_{sub_j}$, $x_{sub_j}$ to $x_{lst_{sub_j}}$, and $x_{sub_j}$ to $V_{sub_j}$, respectively. $N_{x_{sub_j}}x_{sub_j}$ is the fundamental matrix of the local Markov chain of $sub_j$, and $f_{x_{sub_j}}^{t}$ is the known vector of internal nodes in $sub_j$. 

Fig. 3. PDABioMaC algorithm
A. Circuit Partition
To achieve better performance when building the connectivities between internal nodes and cut nodes, the balance partition is needed. Because of the great efficiency (constant proportional to the number of wire segments) and the balance property, the multinode level-set expansion algorithm [10] is adopted to partition the power delivery network.

B. Equivalent Local Markov Chain Model for Each Subnetwork

After partitioning, the equivalent local Markov chain model of each subnetwork can be constructed by performing the absorbing Markov chain relaxation presented in Section II to each subnetwork. For example, the nodal voltages of internal nodes in each $sub_j$ can be represented as

$$V^{t}_{x_{sub_j}} = B_{x_{sub_j}\rightarrow V}V^t_{V_{sub_j}} + B_{x_{sub_j}\rightarrow x_{lst_{sub_j}}}V^{t-h}_{x_{sub_j}}$$

Here $V^t_{V_{sub_j}}$, $V^{t-h}_{x_{sub_j}}$, and $V^{t-h}_{x_{sub_j}}$ are state vectors of $x_{sub_j}$ at time $t$, $x_{sub_j}$ at time $t-h$, unknown nodal voltages of $x_{sub_j}$, and $V^{t-h}_{x_{sub_j}}$, respectively. $B_{x_{sub_j}\rightarrow V}$ and $B_{x_{sub_j}\rightarrow x_{lst_{sub_j}}}$ and $B_{x_{sub_j}\rightarrow V}$ are absorption probability matrices from $x_{sub_j}$ to $V_{sub_j}$, $x_{sub_j}$ to $x_{lst_{sub_j}}$, and $x_{sub_j}$ to $V_{sub_j}$, respectively. $N_{x_{sub_j}}x_{sub_j}$ is the fundamental matrix of the local Markov chain of $sub_j$, and $f_{x_{sub_j}}^{t}$ is the known vector of internal nodes in $sub_j$. 

Fig. 4. Definitions of terms and symbols.
Since the boundary nodes are also the internal nodes, the boundary nodal voltages of subj can be written as

\[ V_{b_i} = B_{b_i}^{-1}e_i + V_{e_i}^{t-h} + N_{b_i}^{-1}f_i \]

(8)

where the definitions of all matrices and vectors are similar to Equation (7) with corresponding subscripts and superscripts.

C. Equivalent Global Markov Chain Model for All Cut Nodes

Fig. 4. The topology of global absorbing Markov chain. Each solid circle is a non-absorbing state of cut nodes, and each rectangle is a subnetwork.

The equivalent global Markov chain model can be built to represent the relationships between all cut nodes on cut lines and the boundary nodes of all subnetworks. Similar to Equation (4), the state vector of the unknown nodal voltages of all cut nodes at time \( t \) can be formulated as

\[ V_{i}^{t} = Q_{i}^{-1}V_{c}^{t} + Q_{i}^{-1}V_{b}^{t} + R_{i}e^{-v}V_{i}^{t-h} + P_{i}V_{e}^{t} + f_{i} \]

(9)

where \( Q_{i}^{-1} \) and \( Q_{i}^{-1} \) are transition probability matrices between cut nodes, and from cut nodes to boundary nodes, respectively. \( V_{b}^{t-h} \) and \( V_{b}^{t-h} \) are state vectors of boundary nodal voltages consisting of all \( V_{b}^{t-h} \)'s, and \( V_{b}^{t-h} \)'s, respectively. The definitions of rest matrices and vectors are similar to Equation (4). Each subscript, ‘c’, is the set of all cut nodes, and the suffix, ‘c’, means on the cut lines.

According to the topology of cut nodes in Fig. 4, (I - \( Q_{i}^{-1} \)) is nearly tridiagonal. Therefore, it can be solved by performing a LU factorization of \( (I - Q_{i}^{-1}) \) without sacrificing the efficiency. Substituting \( Z_{m=0}^{\infty}Q_{i}^{-1} = (I - Q_{i}^{-1}) \) into Equation (9) as \( n \rightarrow \infty \), \( V_{c}^{t} \) is equal to

\[ V_{c}^{t} = B_{c}e_{c}V_{b}^{t} + B_{c}e_{c}V_{b}^{t-h} + N_{c}^{-1}e_{c} \]

(10)

Definitions of all symbols in (10) are similar to Equations (7) and (9) with corresponding subscripts and superscripts.

The \( V_{c}^{t} \) in Equation (8) is a function of \( V_{e}^{t} \), and \( V_{c}^{t} \) in Equation (10) is a function of \( V_{b}^{t} \). This bipartite property inspires us to develop the “bipartite absorbing Markov chain relaxation” by integrating those local Markov chains and the global Markov chain to evaluate all unknown nodal voltages.

D. Bipartite Absorbing Markov Chain Relaxation Nodal Voltage Solver

Based on the observation from the last subsection, our bipartite absorbing Markov chain relaxation is summarized in Fig. 5. When executing these LU-decompositions, the modified minimum degree ordering algorithm [11] is used for preserving the sparse property of local-nets. The computational equations of steps 3(a) and 3(c) in Fig. 5 are derived from Equations (7) and (10), and \( f_{c} \) and \( f_{c} \) are corresponding rest known value vectors. Moreover, the SOR (successive over relaxation) [10] is utilized to further speed up the convergent rate of our bipartite absorbing Markov chain relaxation. The following theorem guarantees the stability of this algorithm.

**Theorem:** The convergence of bipartite absorbing Markov chain relaxation is guaranteed.

Moreover, this algorithm always achieves an accurate solution within two iterations for RC circuits in our simulation.

IV. RLKC CIRCUITS ANALYZING

The prototype of PDAIBMuc presented in section III can be combined with our proposed MOR (model order reduction) technique to deal with the issue of mutual coupling effects.

![Model of RLKC wires](image)

Since the optimal equivalent RLKC circuit is shown in Fig. 6(a). For clarity, it does not show the part of mutual inductances. By applying the KCL at an arbitrary node \( x \), its nodal equation can be written as

\[ v(x) + c_{x} \frac{dv(x)}{dt} = \sum_{j \in \partial(x)} p_{xj} v(x) - i_{xj} - \sum_{j \in \partial(x)} g_{xj} v(x) \]

(11)

where \( p_{xj} \) is the current source at node \( x \), \( c_{x} \) is the gate or decoupling capacitance attached at node \( x \), \( g_{xj} \) is the current and \( v(x) \) is the voltage across the inductance \( i_{xj} \). 

V(x) = \frac{i_{xj}}{\sum_{j \in \partial(x)} g_{xj}} \times \sum_{j \in \partial(x)} g_{xj} \times \sum_{j \in \partial(x)} g_{xj}

(12)

where \( i_{xj} \) is the branch current across the inductance \( i_{xj} \), \( i_{xj} \) is the current device at node \( x \), \( c_{x} \) is the gate or decoupling capacitance attached at node \( x \), \( g_{xj} \) is the current, and \( v(x) \) is the voltage across the inductance. The d = 0 when node x is connected to the inductance, and d = 1 when node x is connected to the resistance.

By integrating all equations of nodes into a set of system equations, and taking the Laplace transform on it, the matrix form of this system in Laplace domain is

\[ V(s) + sC V(s) = \hat{Q} V(s) - \hat{I} V(s) + \hat{G}_L V_L(s) \]

(12)

where \( \hat{C} = diag(c_1, c_2, \ldots, c_n) \) is a diagonal matrix, \( \hat{G}_L \) is a coefficient matrix of \( V_L(s) \) with each entry \( x, j \) being equal to \( g_{xj} \), and \( Q \) is a matrix of transition probabilities with each entry \( x, j \) being equal to \( p_{xj} \) and its sum of each row is one. \( V_L(s) \), and \( V(s) \) are vectors of branch voltages across the inductances, and nodal voltages, respectively, with each entry being the desired voltage \( v(x) \).

Since we need \( V_L(s) \) to obtain \( V_L(s) \) in equation (12), we need the computational equations for \( V_L(s) \). Taking Fig. 6(b) as an example, \( V_L(s) \) can be obtained by

\[ G_R (V_R(s) - V_M(s)) = \hat{I}_L(s) \]

(13)

\[ V_L(s) = L_{d} \hat{I}_L(s) \]

(14)

\[ V_M(s) - V_K(s) = \hat{V}_L(s) \]

(15)
Fig. 7. Runtime of RC and RLKC circuits. The ‘−’ means that the methodology is failed because it is out of memory.

1 Begin
2 \[ \mathbf{m}_t = \mathbf{Q} \mathbf{m}_0 - \mathbf{I}_s; \mathbf{m}_t = 0. \]
3 For \( q = 1 \) to the desired order 
4 \[ \mathbf{m}_{L_q} = \mathbf{L}_R \mathbf{m}_{R_{q-1}} - \mathbf{m}_{R_{q-1}} - \mathbf{m}_{L_{q-1}} - \mathbf{I}_q. \]
5 \[ \mathbf{m}_q = \mathbf{Q} \mathbf{m}_q + \mathbf{G}_L \mathbf{m}_{I_q} - \mathbf{C} \mathbf{m}_{E_q} - \mathbf{I}_q. \]
6 End
7 Ortho-normalize \( \mathbf{m}_q \) and \( \mathbf{m}_{E_q} \) for all \( q \).
8 Create the reduced system for original power delivery network via the congruent transformation.
9 End

Fig. 8. Algorithm of PDA\textit{BiMaC} for RLKC circuit analysis.

where \( \mathbf{V}_R(s) \) is the vector of nodal voltages with their nodes connecting to the resistances of wire segments (i.e. each \( v_{R_{q}}(s) \)). \( \mathbf{V}_M(s) \) is the vector of nodal voltages with their nodes being the middle nodes of wire segments (i.e. each \( v_{m_q}(s) \)). \( \mathbf{V}_K(s) \) is the vector of branch voltages across the inductances of wire segments (i.e. each \( v_{L_{q}}(s) \)). \( \mathbf{I}_q \) is the vector of nodal currents through inductances (i.e. each \( I_{L_{q}}(s) \)). \( \mathbf{L} \) is the inductance matrix [12], and \( \mathbf{G}_R \) is the coefficient matrix with entries corresponding to conductors of wire segments (i.e. each \( q \)).

Each \( \mathbf{V}(s) \), \( \mathbf{I}(s) \), and \( \mathbf{V}_L(s) \) can be expanded into a Taylor series around \( s = 0 \) as:

\[
\mathbf{V}(s) = \sum_{q=1}^{\infty} \mathbf{s}^q \mathbf{I}_q \mathbf{s}^q; \mathbf{V}(s) = \mathbf{m}_s \mathbf{s}^q, (16)
\]

where \( \mathbf{m}_q \), \( \mathbf{I}_q \), and \( \mathbf{m}_{L_q} \) are \( q \)-th moment vectors of \( \mathbf{V}(s) \), \( \mathbf{I}(s) \), and \( \mathbf{V}_L(s) \), respectively. Each \( \mathbf{I}_q \) can be calculated by using the technique proposed in [2].

Combining Equations (12)~(16), our proposed MOR technique is summarized in Fig. 8. The advantage of above moment recurrence is that the equations for calculating \( \mathbf{m}_q \) and \( \mathbf{m}_{L_q} \) are decoupled, and \( \mathbf{m}_{L_q} \) can be got by directly performing the matrix-vector multiplication. Moreover, since \( Q \) is a transition probability matrix, forms of \( Q_{12} \) and \( Q_{21} \) can be got by the bipartite Markov chains. Hence, \( \mathbf{m}_q \) and \( \mathbf{m}_{L_q} \) can be solved by the prototype of our PDA\textit{BiMaC}.

V. EXPERIMENTAL RESULTS

In this section, we demonstrate the efficiency and accuracy of our PDA\textit{BiMaC} by comparing our results with two state-of-the-art methods. The test circuits are randomly generated networks which consist of lumped RC/RLKC segments and many current sources. The parameter values are generated by the standard 0.13\( \mu \text{m} \) process (R: 0.046\( \Omega /\mu \text{m} \), L: 1.69\( \mu \text{H}/\mu \text{m} \), C: 0.13011\( /\mu \text{m} \)), and the supply voltage is 1 volt. The partial inductance matrix \( \mathbf{L} \) is estimated by the method proposed in [1], and the reluctance matrix \( \mathbf{K} \) is constructed by the method proposed in [12] with the window size being 10\( \times \)10. The current sources are modeled as piecewise linear waveforms and their maximum amplitude is 3.2\( m\text{A} \).

PDA\textit{BiMaC} is implemented in C++ language and tested on a Pentium IV 3.4-GHz machine with 3 GB memory.

The accuracy of PDA\textit{BiMaC} is demonstrated by comparing the result with an accurate time domain solver, InductWise [1]. The comparing errors of all circuit are listed in the last two columns of Fig. 7. These results show that the maximum error of PDA\textit{BiMaC} is less than 1.43\% for each RC circuit, and the maximum error of PDA\textit{BiMaC} is less than 0.034\% for each RLKC circuit, and the maximum error of PDA\textit{BiMaC} is less than 2.78\% and its average error is less than 0.091\% for each RLKC circuit. These results show the ability of PDA\textit{BiMaC} for analyzing the power delivery networks with considering the effects of mutual inductances.

To show the robustness and efficiency of PDA\textit{BiMaC}, the DC analysis and transient simulation with 70 time steps are executed for each RC/RLKC circuit, and our runtimes are compared with two efficient methods, InductWise [1] and IEKS [2]. The convergence criterion is the maximum voltage difference being less than \( 10^{-3} \). The results are also shown in Fig. 7. The significant speed improvement, over 100 times faster than InductWise and over 10 times faster than IEKS, and less memory usage are observed.

VI. CONCLUSION

A Markov chain based hierarchical solver, PDA\textit{BiMaC}, for the power delivery network has been presented. This engine integrates the local and global Markov chains by the bipartite Markov chain relaxation to improve the efficiency and save the memory usage for analyzing power delivery networks. The significant speed improvement and the less memory usage have been demonstrated in the experimental results.

REFERENCES