Chap 04 Circuit Theorems
Outline

• Introduction
• Linearity Property
• Superposition
• Source Transformations
• Thevenin’s Theorem
• Norton’s Theorem
• Derivations of Thevenin’s and Norton’s Theorems
• Maximum Power Transfer
Introduction

A large complex circuits

Circuit Theorems

- Thevenin’s theorem
- Circuit linearity
- Source transformation

Simplify circuit analysis

- Norton theorem
- Superposition
- Max. power transfer
Linearity Property

Homogeneity property (Scaling)

\[ i \rightarrow v = iR \]
\[ ki \rightarrow kv = kiR \]

Additivity property

\[ i_1 \rightarrow v_1 = i_1R \]
\[ i_2 \rightarrow v_2 = i_2R \]

\[ i_1 + i_2 \rightarrow (i_1 + i_2)R = i_1R + i_2R = v_1 + v_2 \]
A linear circuit is one whose output is linearly related (or directly proportional) to its input.
Linear Circuit (cont.)

- Linear circuits consist of:
  - linear elements
  - linear independent sources
  - linear dependent sources

- Notes:
  - The linear independent sources are considered as the inputs.

- Examples:
  - $v_s = 10\,\text{V} \rightarrow i = 2\,\text{A}$
  - $v_s = 1\,\text{V} \rightarrow i = 0.2\,\text{A}$
  - $v_s = 5\,\text{mV} \leftarrow i = 1\,\text{mA}$

- Power:
  \[ p = i^2 R = \frac{v^2}{R} : \text{nonlinear} \]
Q: Find $I_0$ when $v_s=12\,\text{V}$ and $v_s=24\,\text{V}$.
Example 4.1 (cont.)

KVL at loop 1: \[ 12i_1 - 4i_2 + v_s = 0 \ldots (a) \]

KVL at loop 2: \[-4i_1 + 16i_2 - 3v_x - v_s = 0; (v_x = 2i_1) \]

\[ \Rightarrow -10i_1 + 16i_2 - v_s = 0 \ldots (b) \]

\[(a) \times 5 + (b) \times 6 : \]

\[ 76i_2 - v_s = 0 \Rightarrow i_2 = \frac{v_s}{76} \]

\[
\begin{align*}
\text{When } v_s &= 12V: & I_0 &= i_2 = \frac{12}{76} \text{ A} \\
\text{When } v_s &= 24V: & I_0 &= i_2 = \frac{24}{76} \text{ A}
\end{align*}
\]

Showing that when the source value is doubled, \( I_0 \) doubles.
Example 4.2

Q: Assume $I_0 = 1 \text{A}$ and use linearity to find the actual value of $I_0$.

If $I_0 = 1\text{A}$, then $V_1 = (3 + 5)I_0 = 8\text{V}$

$I_1 = V_1 / 4 = 2\text{A}$, 

$\Rightarrow I_2 = I_1 + I_0 = 3\text{A}$

$V_2 = V_1 + 2I_2 = 8 + 6 = 14\text{V}$, $I_3 = \frac{V_2}{7} = 2\text{A}$
Example 4.2 (cont.)

\[ I_4 = I_3 + I_2 = 5 \text{A} \implies \]

\[ I_0 = 1 \text{A} \implies I_{s}^{I_0=1} = 5 \text{A} \]

\[ I_0 = 3 \text{A} \iff I_s = I_s^{I_0=1} \times 3 = 15 \text{A} \]
Superposition Principle

The voltage across (current through) an element in a **linear circuit** is the **algebraic sum** of the voltages across (currents through) that element due to **each independent source** acting alone.

- **Turn off, killed, inactive source:**
  - independent voltage source: 0 V (**short circuit**)
  - independent current source: 0 A (**open circuit**)

- **Dependent sources are left intact.**
Superposition Steps

- **Steps to apply superposition principle**
  1. **Turn off** all independent sources except one source. Find the output (voltage or current) due to that active source using nodal or mesh analysis.
  2. **Repeat** step 1 for each of the other independent sources.
  3. **Find** the total contribution by **adding** algebraically all the contributions due to the independent sources.
Remarks of Superposition

• Turn off independent sources
  – For the voltage source: Set it voltage to be zero; in the other words, “Short the voltage source”
  – For the current source: Set it current to be zero; in the other words, “Open the current source”.

• Superposition involves more work but on simpler circuits.

• Superposition is not applicable to the effect on power because of the nonlinearity of power.
Example 4.3

Q: Use the superposition theorem to find $v$ in the circuit.

Turn off current source 3A, and use voltage division

$v_1 = \frac{4}{4+8} \times 6 = 2V$

$\Rightarrow v = v_1 + v_2 = 2 + 8 = 10V$

Turn off voltage source 6V, and use current division

$i_3 = \frac{8}{4+8} (3) = 2A$

$\Rightarrow v_2 = 4i_3 = 8V$
Example 4.4

Q: Find $i_o$ in the following circuit using superposition.

Let $i_o = i_o' + i_o''$;

\[
\begin{align*}
\begin{cases}
i_o' & \text{is due to 4-A current source} \\
i_o'' & \text{is due to 20-V voltage source}
\end{cases}
\end{align*}
\]
For $i'_o$ : Turn off 20-V source

Loop 1: \[ i_1 = 4A \cdots (a) \]

Loop 2: \[ -3i_1 + 6i_2 - 1i_3 - 5i'_o = 0 \cdots (b) \]

Loop 3: \[ -5i_1 - 1i_2 + 10i_3 + 5i'_o = 0 \cdots (c) \]

KCL at node 0: \[ i_3 = i_1 - i'_o = 4 - i'_o \cdots (d) \]

Substitute (a) and (d) into (b) and (c)

\[
\begin{align*}
3i_2 - 2i'_o &= 8 \\
i_2 + 5i'_o &= 20
\end{align*}
\Rightarrow \quad i'_o = \frac{52}{16} \quad A
\]
Example 4.4 (cont.)

For $i_o''$: $i_o = -i_o''$; Turn off 4-A source

Loop 4:

$$6i_4 - i_5 - 5i_o'' = 0$$

$$\Rightarrow 6i_4 - 4i_o'' = 0$$

Loop 5:

$$-i_4 + 10i_5 + 5i_o'' - 20 = 0$$

$$\Rightarrow i_4 - 5i_o'' = -20$$

$$\therefore i_o'' = \frac{-60}{17} \text{ A}$$

$$\therefore i_o = i_o' + i_o'' = \frac{52}{16} + \left( -\frac{60}{17} \right) = -\frac{8}{17} = -0.4706 \text{ A}$$
Example 4.5

Q: Use the superposition theorem to find \( i \).

Let \( i = i_1 + i_2 + i_3 \);

\[
\begin{align*}
  i_1 & \text{ is due to 12-V voltage source} \\
  i_2 & \text{ is due to 24-V voltage source} \\
  i_3 & \text{ is due to 3-A current source}
\end{align*}
\]
Example 4.5 (cont.)

For $i_1$: Off 24-V and 3-A sources

\[ i_1 = \frac{12V}{4/(8+4) + 3} = \frac{12V}{6\Omega} = 2\text{A} \]
Example 4.5 (cont.)

For $i_2$: Off 12-V and 3-A sources

\[
\begin{align*}
\text{mesh } a &: \quad 16i_a - 4i_b + 24 = 0 \Rightarrow 4i_a - i_b = -6 \cdots (a) \\
\text{mesh } b &: \quad 7i_b - 4i_a = 0 \Rightarrow i_a = \frac{7}{4} i_b \cdots (b)
\end{align*}
\]

Substitute (b) into (a)

\[
i_2 = i_b = -1 \text{A}
\]
Example 4.5 (con.)

For \( i_3 \): Off 24-V and 12-V sources

\[ \frac{v_2 - v_1}{4} = \frac{v_1}{4} + \frac{v_1}{3} \Rightarrow v_2 = \frac{10}{3}v_1 \cdots (a) \]

\[ 3 = \frac{v_2}{8} + \frac{v_2 - v_1}{4} \Rightarrow 24 = 3v_2 - 2v_1 \cdots (b) \]

Substitute (b) into (a)

\[ \therefore v_1 = 3 \text{ and } i_3 = \frac{v_1}{3} = 1\text{A} \]

Thus,

\[ i = i_1 + i_2 + i_3 = 2 - 1 + 1 = 2\text{A} \]
Source Transformation

- A source transformation is the process of replacing a voltage source $v_s$ in series with a resistor $R$ by a current source $i_s$ in parallel with a resistor $R$, or vice versa.
Source Transformation (cont.)

- Equivalent resistance (source off) \( R \)
- Short-circuit current from \( a \) to \( b \) \( i_{sc} = \frac{v_s}{R} = i_s \)

\[
\begin{align*}
  v_s &= i_s R \\
  i_s &= \frac{v_s}{R}
\end{align*}
\]
The source transformation is also applicable to dependent sources.
\[ v = iR + v_s \]

\[ i = \frac{v}{R} - \frac{v_s}{R} \]
Remarks of Source Transformation

- **Arrow of the current source**
  - positive terminal of voltage source

- **Impossible source transformation**
  - The ideal voltage source \((R = 0)\)
  - The ideal current source \((R = \infty)\)

\[
i_s = \frac{v_s}{R} \quad \text{or} \quad v_s = i_s R
\]
Example 4.6

Q: Use source transformation to find $v_o$. 

The diagram shows a circuit with various resistors and currents, where the goal is to find the voltage $v_o$. The circuit is transformed into different configurations (a), (b), and (c), demonstrating the use of source transformation techniques.
The current division can be utilized in Fig. (c) to get

\[ i = \frac{2}{2 + 8} (2) = 0.4 \text{A} \]

and

\[ v_o = 8i = 8(0.4) = 3.2 \text{V} \]
Example 4.7

Q: Find $v_x$ using source transformation
Example 4.7 (cont.)

KVL to the loop in Fig. (b):

\[-3 + 5i + v_x + 18 = 0 \cdots (a)\]

KVL containing 3V \(\rightarrow\) 1\(\Omega\) \(\rightarrow\) \(v_x\):

\[-3 + 1i + v_x = 0\]

\[\Rightarrow v_x = 3 - i \cdots (b)\]

Substituting \((b)\) into \((a)\), we obtain

\[15 + 5i + 3 = 0 \Rightarrow i = -4.5A\]

Thus,

\[v_x = 3 - i = 7.5V\]
Thevenin’s Theorem

- **Thevenin’s Theorem**: A linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source $V_{Th}$ in series with a resistor $R_{Th}$.

Here,

1. $V_{Th}$ is the open circuit voltage at the terminals.
2. $R_{Th}$ is the input or equivalent resistance at the terminals when the independent sources are turned off.
Thevenin’s Theorem (cont.)

(a) Linear two-terminal circuit

(b) Equivalent circuit with Thevenin’s theorem, where $V_{Th}$ and $R_{Th}$ are the open-circuit voltage and the internal resistance respectively.
Property of Linear Circuits

Linear two-terminal circuit

\[ \text{Slope} = \frac{1}{R_{th}} \]
Thevenin’s Voltage Calculation

- Two circuits are *equivalent* if they have the same voltage-current relation at their terminals.
- $V_{Th} = v_{oc}$: open circuit voltage at $a-b$
Thevenin’s Resistance Calculation

- \( R_{Th} = R_{in} \) (because the two circuits are equivalent.)

\( R_{in} \): the input resistance of the dead circuit at \( a-b \).
- \( a-b \) open circuited
- turn off all independent sources

\[ R_{Th} = R_{in} \]

(b)
CASE 1: the network has no dependent sources.

– Turn off all independent source.

– $R_{TH}$: input resistance of the network looking at the terminals $a-b$

\[ R_{Th} = R_{in} \]
CASE 2: the network has dependent sources

Method 1
– Turn off all independent sources.
– Apply a voltage source \( v_o \) at \( a-b \)

\[
R_{Th} = \frac{v_o}{i_o}
\]

Method 2
– Turn off all independent sources.
– Apply a current source \( i_o \) at \( a-b \)

\[
R_{Th} = \frac{v_o}{i_o}
\]

• If \( R_{Th} < 0 \), the circuit supplies power.
Simplifying Circuits by Thevenin’s Theorem

Thevenin’s Theorem

\[ I_L = \frac{V_{Th}}{R_{Th} + R_L} \]

\[ V_L = R_L I_L = \frac{R_L}{R_{Th} + R_L} V_{Th} \quad \text{(voltage division)} \]
Example 4.8

Q: Find the Thevenin’s equivalent circuit to the left of the terminals $a-b$. Then find the current through $R_L = 6, 16, \text{ and } 36 \, \Omega$. 
Example 4.8 (cont.)

Thevenin resistance $R_{Th}$ calculation: Short 32-V voltage source and open 2-A current source

$$R_{Th} = 4 \ || 12 + 1 = \frac{4 \times 12}{16} + 1 = 4 \Omega$$
Example 4.8 (cont)

Thevenin voltage \( V_{Th} \) calculation: Open \( R_L \)

Method 1: Mesh analysis

\[-32 + 4i_1 + 12(i_1 - i_2) = 0, \quad i_2 = -2A\]

\[\therefore i_1 = 0.5A\]

\[V_{Th} = 12(i_1 - i_2) = 12(0.5 + 2.0) = 30V\]

Method 2: Nodal analysis

\[(32 - V_{Th})/4 + 2 = V_{Th}/12\]

\[\therefore V_{Th} = 30V\]

Method 3: Source transform

請自行推導
Example 4.8 (cont.)

To get $I_L$ for $R_L = 3, 16, \text{ or } 36 \Omega$

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L}$$

- $R_L = 6$: $I_L = \frac{30}{10} = 3 \text{A}$
- $R_L = 16$: $I_L = \frac{30}{20} = 1.5 \text{A}$
- $R_L = 36$: $I_L = \frac{30}{40} = 0.75 \text{A}$
Example 4.9

Q: Find the Thevenin’s equivalent at terminals \(a-b\).
(independent + dependent source case)

To find \( R_{Th} \) : Fig(a)

independent source \( \rightarrow 0 \)
dependent source \( \rightarrow \) intact

\[ v_o = 1 \text{V, } R_{Th} = \frac{v_o}{i_o} = \frac{1}{i_o} \]
Example 4.9 (cont.)

- Loop 1:

\[-2v_x + 2(i_1 - i_2) = 0 \text{ or } v_x = i_1 - i_2\]

But \(-4i_2 = v_x = i_1 - i_2\)

\[\therefore i_1 = -3i_2\]
Example 4.9 (cont.)

Loop 2 and 3:

\[4i_2 + 2(i_2 - i_1) + 6(i_2 - i_3) = 0\]

\[6(i_3 - i_2) + 2i_3 + 1 = 0\]

Solving these equations \(i_3 = -1/6\) A.

But \(i_o = -i_3 = \frac{1}{6}\) A

\[\therefore R_{Th} = \frac{1V}{i_o} = 6\Omega\]
To get $V_{Th}$: Fig(b)

Mesh analysis:

$i_1 = 5$

$-2v_x + 2(i_3 - i_2) = 0$

$\Rightarrow v_x = i_3 - i_2$

$4(i_2 - i_1) + 2(i_2 - i_1) + 6i_2 = 0$

$\Rightarrow 12i_2 - 4i_1 - 2i_3 = 0$

But $4(i_1 - i_2) = v_x$

$\therefore i_2 = \frac{10}{3}$.

$V_{Th} = v_{oc} = 6i_2 = 20V$
Example 4.10

Q: Determine the Thevenin’s equivalent circuit.
(dependent source only case)

\[ V_{Th} = 0 \quad R_{Th} = \frac{V_o}{i_o} \]

Nodal analysis:

\[ i_o + i_x = 2i_x + \frac{V_o}{4} \]

\[ \Rightarrow i_0 = i_x + \frac{V_o}{4} \]
Example 4.10 (cont.)

But

\[ i_x = \frac{0 - v_o}{2} = -\frac{v_o}{2} \]

\[ i_o = i_x + \frac{v_o}{4} = -\frac{v_o}{2} + \frac{v_o}{4} = -\frac{v_o}{4} \quad \text{or} \quad v_o = -4i_o \]

Thus \( R_{Th} = \frac{v_o}{i_o} = -4\Omega \): Supplying power
Example 4.10 (cont.)
Example 4.10 (cont.)

(d)
Norton’s Theorem

• **Norton’s theorem** states that a linear two-terminal circuit can be replaced by equivalent circuit consisting of a current source $I_N$ in parallel with a resistor $R_N$.

Here,

- $I_N$ is the short-circuit current through the terminals
- $R_N$ is the input or equivalent resistance at the terminals when the independent sources are turn off.
Norton’s Theorem (cont.)

(a) Linear two-terminal circuit

(b) \( I_N \) \( R_N \)
Norton’s Resistance and Current

- Thevenin and Norton resistances are equal:

\[ R_N = R_{Th} \]

- Short circuit current from \( a \) to \( b \):

\[ I_N = i_{sc} = \frac{V_{Th}}{R_{Th}} \]
Thevenin or Norton Equivalent Circuit

- The open circuit voltage $v_{oc}$ across terminals $a$ and $b$
- The short circuit current $i_{sc}$ at terminals $a$ and $b$
- The equivalent or input resistance $R_{in}$ at terminals $a$ and $b$ when all independent sources are turned off.

$$V_{Th} = v_{oc}$$
$$I_N = i_{sc}$$
$$R_{Th} = \frac{v_{oc}}{i_{sc}} = R_N$$
Example 4.11 (cont.)

Q: Find the Norton equivalent circuit.
To find $R_N$

$$R_N = 5 \|(8 + 4 + 8)$$

$$= 5 \| 20 = \frac{20 \times 5}{25} = 4 \Omega$$
Example 4.11 (cont.)

To find $I_N$

Short circuit between terminals $a$ and $b$.

Mesh Analysis: $i_1 = 2A$, $20i_2 - 4i_1 - 12 = 0$

$\Rightarrow i_2 = 1A = i_{sc} = I_N$
Example 4.11 (cont.)

Alternative method for $I_N$

\[ I_N = \frac{V_{Th}}{R_{Th}} \]

$V_{Th}$: open circuit voltage across terminals $a$ and $b$

Mesh analysis

Loop 1: $i_3 = 2A$

Loop 2: $25i_4 - 4i_3 - 12 = 0$

$\Rightarrow i_4 = 0.8A$

$\therefore V_{oc} = V_{Th} = 5i_4 = 4V$
Example 4.11 (cont.)

Hence,

\[ I_N = \frac{V_{Th}}{R_{Th}} = \frac{4}{4} = 1 \text{A} \]
Example 4.12

Q: Using Norton’s theorem, find $R_N$ and $I_N$ at terminals $a-b$. 
Example 4.1

To find $R_N$

4Ω resistor shorted $\Rightarrow i_x = 0$

$$i_o = \frac{1}{5} = 0.2\text{A}$$

$$\therefore R_N = \frac{v_o}{i_o} = \frac{1}{0.2} = 5\Omega$$
To find $I_N$

\[ 4\Omega \ || 10\Omega \ || 5\Omega \ || 2i_x : \text{Parallel} \]

\[ i_x = \frac{10 - 0}{4} = 2.5\text{A}, \]

\[ i_{sc} = \frac{10}{5} + 2(i_x) = \frac{10}{5} + 2(2.5) = 7\text{A} \]

\[ \therefore I_N = 7\text{A} \]
• The Thevenin’s and Norton’s theorems can be proved by using the *superposition principle*. 
Derivation of Thevenin’s Theorem

- For simplicity, suppose the linear circuit contains four independent voltage sources, \( v_{s1}, v_{s2}, i_{s1} \) and \( i_{s2} \).
- By superposition, \( v = A_0i + A_1v_{s1} + A_2v_{s2} + A_3i_{s1} + A_4i_{s2} \cdots (a) \)
- Rewrite \( v \) to be \( v = A_0i + B_0 \)
- Here, \( B_0 = V_{Th} \) because \( B_0 \) is calculated as \( i = 0 \) (terminals \( a \) and \( b \) are opened.)
- Turn off all internal sources, \( B_0 = 0 \), \( \Rightarrow v = A_0i = R_{eq}i = R_{TH}i \)
- Finally, \( (a) \) becomes \( v = R_{TH}i + V_{TH} \)
Derivation of Norton’s Theorem

- The $i$ can be expressed by superposition as $i = C_0v + D_0 \cdots (a)$
- Here, $C_0v$ is contributed by $v$, and $D_0$ is contributed by all internal independent sources.
- $D_0 = -i_{sc} = -I_N$ because $D_0$ is calculated as $v=0$ (terminals $a$ and $b$ are shorted.)
- Turn off all internal independent sources, $D_0 = 0$,
  
  $$i = C_0v = G_{eq}v = \left(1/R_{eq}\right)v = \left(1/R_{TH}\right)v$$
- Finally, $(a)$ becomes
  $$i = \frac{v}{R_{TH}} - I_N$$
Maximum Power Transfer

\[ p = i^2 R_L = \left( \frac{V_{TH}}{R_{TH} + R_L} \right)^2 R_L \]
Maximum Power Theorem

- **Maximum power** is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{TH}$).
Proof of Maximum Power Theorem

\[ p = \left( \frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L \]

\[ \frac{dp}{dR_L} = V_{Th}^2 \left( R_{Th} + R_L \right)^2 - 2R_L \left( R_{Th} + R_L \right) \left( R_{Th} + R_L \right)^4 \]

\[ = V_{Th}^2 \frac{\left( R_{Th} + R_L - 2R_L \right)}{\left( R_{Th} + R_L \right)^3} = 0 \]

Set \( \frac{dp}{dR_L} = 0 \Rightarrow R_L = R_{Th} \)

\[ p_{\text{max}} = \frac{V_{Th}^2}{4R_{Th}} \]
Example 4.13

Q: Find the value of $R_L$ for maximum power transfer in the circuit. Find the maximum power.
Example 4.1

\[ R_{Th} = 2 + 3 + 6\|12 = 5 + \frac{6 \times 12}{18} = 9\Omega \]
Example 4.13 (cont.)

Loop 1: \(-12 + 18i_1 - 12i_2 = 0\)

Loop 2: \(i_2 = -2\,A\)

\(\Rightarrow i_1 = -\frac{2}{3}\,A\)

To get \(V_{th}\) by applying KVL around Loop 3:

\(-12 + 6i_i + 3i_2 + 11(0) + V_{Th} = 0 \Rightarrow V_{Th} = 22\,V\)

Set \(R_L = R_{Th} = 9\,\Omega\) \(\Rightarrow p_{max} = \frac{V_{Th}^2}{4R_L} = \frac{22^2}{4 \times 9} = 13.44\,W\)
Homework (Due day: 10/29/2009)

- Problems 4.3, 4.6, 4.8, 4.15, 4.18, 4.22, 4.26, 4.40, 4.42, 4.43, 4.55, 4.60, 4.67, and 4.75.